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Construction of Lyapunov Functions for Nonlinear Systems Using Normal Forms*

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In [8], the authors used normal form theory to construct Lyapunov functions for critical nonlinear systems in normal form coordinates. In this work, the authors expand on those ideas by providing a method for constructing the associated normal form transformations that gives rise to the systematic development of a method for constructing Lyapunov functions for critical nonlinear systems in their original coordinates.

1. INTRODUCTION

Lyapunov functions are useful for determining the stability properties of nonlinear systems. This paper presents a method for constructing Lyapunov functions for locally asymptotically stable critical nonlinear systems that is both simple and systematic. Normal form theory will be used to construct the Lyapunov functions, and an important contribution of this work is the provision of a method for systematically computing the transformations that put critical nonlinear systems into an approximate normal form.

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Without loss of generality, the systems investigated in this paper are of the form

$$\dot{x}_c = A^0 x_c + f(x_c, x_s), \tag{1}$$

$$\dot{x}_s = A^- x_s + g(x_c, x_s), \tag{2}$$

where $x_c \in R^{n_0}$, $x_s \in R^{n_-}$. The vector-valued functions $f(\cdot)$ and $g(\cdot)$, which are assumed to be differentiable to any desired order, vanish at the origin ($x_c = 0, x_s = 0$), along with their first-order derivatives. All of the eigenvalues of A^0 are critical (have zero real parts), while all of the eigenvalues of A^- are stable (have negative real parts). Throughout this paper, the origin ($x_c = 0, x_s = 0$) is assumed to be the equilibrium point of interest.

The local stability analysis of the system (1), (2) can be reduced to the study of that of an n_0-dimensional purely critical subsystem using results from center manifold theory [2]. Due to the limited availability of local stability tests for purely critical nonlinear systems [7, 8], this work will focus on the construction of Lyapunov functions for two classes of critical nonlinear systems:

Case 1. Nonlinear systems of the form (1), (2) with $A^0 = 0$, a scalar.

Case 2. Nonlinear systems of the form (1), (2) with the purely critical 2×2 matrix A^0 having a pair of purely imaginary eigenvalues. Without loss of generality, A^0 will be assumed to have the form, $A^0 = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$ with $\omega > 0$.

In general, it is not easy to construct a Lyapunov function for a critical nonlinear system that is known to be locally asymptotically stable (LAS). In [8], a procedure for constructing Lyapunov functions for LAS Case 1 and Case 2 critical nonlinear systems was developed. In that work, once a transformation had been applied to the system's original coordinates to put the system into an approximate normal form, a Lyapunov function could be constructed for a Case 1 or Case 2 critical system in the new set of coordinates. The problem then remained to find a systematic method for constructing the normal form transformations. A solution to this problem is presented in this work.

In [5], Fu and Abed also proposed an approach for directly constructing Lyapunov functions in the original coordinates for these classes of critical LAS nonlinear systems. Their approach relied on a result about local definiteness of a class of scalar bivariate functions. In that work, Lyapunov functions were constructed explicitly only when the lowest (first, second, and third) order terms in the Taylor series expansion of the nonlinear dynamics were sufficient to decide the local asymptotic stability (LASY). That work cannot be directly applied to those nonlinear systems for which higher than third order nonlinearities must be taken into consideration in order to conclude LASY. The work presented here is much less restrictive and represents an improvement upon previous results. The method for constructing Lyapunov functions that is developed in this work is based on normal form theory and uses properties of homogeneous expansions of smooth maps for constructing normal form transformations. The main contribution of this work is the computation of a transformation that takes a critical system to an approximate normal form. This procedure involves appropriate choices of the complementary spaces that comprise the normal form decomposition and observations about the form of the transformations that take a critical system to its approximate normal form.

The organization of this paper is as follows. Normal forms are discussed in Section 2. In Section 3, a systematic method for computing normal forms and the associated smooth transformations is presented. Section 4 gives a general procedure for constructing Lyapunov functions for critical nonlinear systems, as well as a procedure for the construction of Lyapunov functions for Case 1 and Case 2 smooth critical nonlinear systems in their original coordinates. A brief summary in Section 5 concludes the paper.

2. SOME DETAILS ABOUT NORMAL FORMS

For $m \geq 0$, define $H(n,m,k)$ to be the linear space generated by the homogeneous maps of degree k from $R^n \to R^m$. The inner product defined in [1] can be ascribed to $H(n,m,k)$.

For a given $n \times n$ matrix, A, the Hilbert space $H(n,n,k)$ can be decomposed with respect to the range $R(n,n,k)$ of the linear operator

$$\begin{aligned}\left(L_A^k \phi_k\right)(y) &= \left[\phi_k(y), Ay\right] = -\left[Ay, \phi_k(y)\right] \\ &= D\phi_k(y)Ay - A\phi_k(y), \qquad \phi_k(\cdot) \in H(n,n,k),\end{aligned}$$

and a (nonuniquely defined) complement $C(n,n,k)$ in $H(n,n,k)$. Here $D\phi_k(y) = \partial\phi_k(y)/\partial y$ is the Jacobian of $\phi_k(y)$.

One such decomposition of $H(n,n,k)$ is an orthogonal decompostion given by

$$H(n,n,k) = R(n,n,k) \oplus \operatorname{Ker}\left(L_{A^T}^k\right),$$

where $\operatorname{Ker}(L_{A^T}^k)$ is the nullspace of $L_{A^T}^k$ [1]. This is the complementary space that will be used to develop normal forms in this work, although the development given in this paper will hold for any complementary space.

With this Hilbert space decomposition in hand, the concept of the normal form, along with its dynamics-simplifying properties, is ready to be introduced. Without loss of generality, consider an analytic nonlinear dynamical system expanded as a series of homogeneous polynomial maps as follows:

$$\begin{aligned}\dot{x} = f(x) &= f^1(x) + f^2(x) + \cdots + f^{k-1}(x)\\ &\quad + f^k(x) + f^{k+1}(x) + \cdots, \qquad x \in R^n, \end{aligned}\tag{3}$$

where $f^i(\cdot) \in H(n, n, i)$. Here $f^1(x) = Ax$ with $A = Df(0)$.

For $k \geq 2$, consider the effect of a (near identity) change of coordinates of the form

$$x = y + P^k(y), \qquad P^k(\cdot) \in H(n, n, k). \tag{4}$$

Differentiating both sides of (4) with respect to time and using (3) yields

$$\left(I + DP^k(y)\right)\dot{y} = f\left(y + P^k(y)\right). \tag{5}$$

Note that $(I + DP^k(y))$ is always invertible in a sufficiently small neighborhood of the origin, and

$$\left(I + DP^k(y)\right)^{-1} = I - DP^k(y) + O\left(|y|^{2k-2}\right), \tag{6}$$

where $O(|y|^{2k-2})$ denotes linear combinations of terms of degree $(2k - 2)$ or higher.

Equations (5), (6) and (3) yield the dynamics:

$$\begin{aligned}\dot{y} = g(y) &= g_k^1(y) + g_k^2(y) + \cdots + g_k^{k-1}(y)\\ &\quad + \overbrace{f^k(y) - \left[Ay, P^k(y)\right]} + g_k^{k+1}(y) + \cdots, \end{aligned}\tag{7}$$

where $g_k^i(\cdot) = f^i(\cdot)$ for $i = 1, 2, \ldots, k - 1$. For $i \geq k + 1$, $g_k^i(\cdot)$ is not in general equal to $f^i(\cdot)$ in (3). Here the subscript k is used to denote the k-dependence of the transformation $P^k \in H(n, n, k)$. The kth order term in (7) is given by

$$g_k^k(y) = f^k(y) - \left[Ay, P^k(y)\right] = f^k(y) - \left(L_A^k P^k\right)(y). \tag{8}$$

Equation (8) suggests the possibility of choosing P^k in (4) so as to use the properties of the linear operator L_A^k: $H(n, n, k) \to H(n, n, k)$ for each $k \geq 2$ to simplify the dynamics, (3).

If $f^k \in R(n, n, k)$, $P^k(\cdot)$ may be chosen such that $(L_A^k P^k)(y) = f^k(y)$, and, consequently, there would be no terms of degree k in (7). In the more general cases where $f^k \notin R(n,n,k)$, $P^k(\cdot)$ may be chosen such that $g_k^k(y) = f^k(y) - [Ay, P^k(y)]$ is in the $C(n,n,k)$, the complementary space. The remaining terms of degree k or less in (7) can be considered, in some sense, to be essential in determining the dynamical behavior of (3) near the origin.

The transformation described above applied iteratively will yield the normal form transformation. This method for constructing the normal form transformation involves the iterative application of a sequence of changes of coordinates (4) to the system (3), starting with $k = 2$. At each step of the iteration, P^k is chosen such that $g_{k-1}^k(y) - [Ay, P^k(y)] \in \mathrm{Ker}(L_{A^T}^k)$. After a finite number of such iterations, the system (3) is said to be put into an approximate normal form. Clearly, the process of normalization is intrinsically local in the sense that the change of coordinates is only valid in some neighborhood of the origin. The normal forms in the following theorem are obtained in the manner described in this discussion.

THEOREM 1 (Normal Form Theorem) [1, 2]. *For the system* (3), *given the sequence of decompositions* (2) *and any finite positive integer* k, *there exists a series of changes of coordinates*, $x_{i-1} = x_i + P^i(x_i)$, $x_1 = x$, $x_k = y$, $P^i \in H(n,n,i)$, $i = 2,3,\ldots,k$, *and a neighborhood* Ω *of the origin*, *such that*, *for* $y \in \Omega$, *the system* (3) *is transformed to a system of the form*

$$\dot{y} = g^1(y) + g^2(y) + \cdots + g^k(y) + O\left(\|y\|^{k+1}\right), \tag{9}$$

where $g^1(y) = Ay$ *and* $g^i(y) \in \mathrm{Ker}(L_{A^T}^i)$ *for* $k = 2,3,\ldots,k$.

Remark 2. Clearly, the structure of the normal form of the system (3), as well as its actual expression given by (9), depends on A. Even if A is fixed, the normal form is not unique, but depends on the choice of the sequence of complementary subspaces $C(n,n,i)$, as well as other factors.

Remark 3. As long as the number of iterations of the changes of coordinates is finite, the normal form of the system (3) given by (9) must be convergent in some neighborhood Ω of the origin. The limiting result of this iteration may not be convergent. In the case that the resulting normal form process is convergent, and $f \in C^\omega$, the resulting full normal form has an analytic expansion of the form:

$$\dot{y} = g(y) = g^1(y) + g^2(y) + g^3(y) + \cdots, \tag{10}$$

where $g^1(y) = Ay$ and $g^k \in \mathrm{Ker}(L_{A^T}^k)$ for $k = 2,3,\ldots$.

Remark 4. The structure of an approximate normal form for a critical nonlinear system is such that the critical dynamics are locally decoupled from the stable dynamics, in that all the lowest-order terms in the critical dynamics will be independent of the stable dynamics [9]. This feature of the normal forms of critical systems is what lends them so well to stability analysis of critical nonlinear systems.

3. COMPUTING NORMAL FORMS

In this section, efficient algorithms for computing normal forms and the associated transformations are introduced. This will be accomplished in Section 3.2, using a method presented in Section 3.1 for relating the dynamics of systems that are connected by smooth transformations that are not necessarily normal form transformations.

3.1. *General Algorithms for the Transformations*

Without loss of generality, an artificial parameter a will be introduced to take advantage of the properties of homogeneous maps. Consider a nonlinear dynamical system depending on a parameter as follows:

$$\dot{x} = f(x, a), \tag{11}$$

where $f\colon R^n \times R \to R^n$ and $0 \neq a \in R$.

Let

$$x = u(y, a) \tag{12}$$

be a transformation (henceforth called the backward transformation) with $u(y, 0) = y$, with its inverse (henceforth called the forward transformation) given by

$$y = v(x, a). \tag{13}$$

Note that $v(x, 0) = x$.

If the transformation (12) is applied to the dynamical system (11), then the dynamics for y are given by

$$\dot{y} = g(y, a) \triangleq \frac{\partial v(x, a)}{\partial x} f(x, a)\bigg|_{x = u(y, a)}. \tag{14}$$

It is desirable to have efficient algorithms to derive (14) from (11) using (12) and/or (13). To simplify the description of the algorithms to be derived, an artificial function $U(u, a)$, which is directly related to $u(y, a)$, is introduced via the following partial differential equation (PDE):

$$\frac{\partial u(y, a)}{\partial a} = U(u, a), \qquad u(y, 0) = y. \tag{15}$$

Let $u(y, a)$ be the (assumed to be unique) solution to the PDE. The backward transformation is then constructed as $x = u(y, a)$. Later it will be shown that, for any given function $u(y, a)$, there exists a unique function $U(u, a)$ satisfying (15) and vice versa. An algorithm for computing $U(u, a)$ from a given $u(y, a)$ or for computing $u(y, a)$ from a given $U(u, a)$ will also be given later (in Procedure 1).

Using (15), an algorithm for deriving (14) from (11) is given in the following theorem.

THEOREM 5 [4]. *For u given by* (15), *let $f(x, a)$, $g(y, a)$, and $U(u, a)$ in* (11), (14), *and* (15) *be expanded in a homogeneous series as follows*:

$$f(x,a) = \sum_{m=0}^{\infty} f_m(x) \frac{a^m}{m!}, \tag{16}$$

$$g(y,a) = \sum_{m=0}^{\infty} g_m(y) \frac{a^m}{m!}, \tag{17}$$

$$U(u,a) = \sum_{m=0}^{\infty} U_m(u) \frac{a^m}{m!}. \tag{18}$$

The sequence $\{g_i\}$ in (17) *satisfies*

$$g_i = f_0^{(i)}, \tag{19}$$

where, for $i = 0, 1, 2, \ldots,$ the sequence $f_{i-m}^{(m)}$ is defined recursively via

$$f_i^{(0)} = f_i, \tag{20}$$

$$f_{i-m}^{(m)} = f_{i-m+1}^{(m-1)} + \sum_{j=0}^{i-m} \binom{i-m}{j} \left[U_j, f_{i-m-j}^{(m-1)}\right], \qquad m = 1, 2, \ldots, i. \tag{21}$$

Here

$$\binom{i-m}{j} = \frac{(i-m)!}{j!(i-m-j)!}$$

is the binomial coefficient and

$$[f, g](x) = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

is the Lie bracket.

Remark 6. In practice, only the first few terms in the Taylor series of f (or u, v, g, and U, respectively) in a are important in stability analysis. However, it is convenient to develop these algorithms while considering the infinite power series in a, even though the functions $f, u, v, g,$ and U will not necessarily have a convergent power series in a. The use of this formalism presents no difficulty as long as only a finite number of terms of the series are actually calculated.

The first few elements in the sequence $\{g_m\}$ can be written as follows:

$$\begin{aligned}
g_0 &= f_0,\\
g_1 &= f_1 + [U_0, f_0] = f_1 - [f_0, U_0],\\
g_2 &= f_2 + [U_0, f_1] + [U_0, g_1] - [f_0, U_1],\\
g_3 &= f_3 + [U_0, f_2] + 2[U_1, f_1] + [U_0, f_2 + [U_0, f_1] + [U_1, f_0]]\\
&\quad + [U_1, g_1] + [U_0, g_2] - [f_0, U_2],
\end{aligned}$$

and, in general,

$$g_m = \Gamma_m - [f_0, U_{m-1}], \tag{22}$$

where Γ_m is a map depending only on $f_0, f_1, \dots, f_m$ and $U_0, \dots, U_{m-2}$.

Notice the similarity between (8) and (22). In fact, it is this dependency that will give rise to a choice of U_m so that $\{g_m\}$ represents the normal form dynamics.

The next theorem is useful in deriving an explicit expression of $u(y, a)$ when the sequence $\{U_m\}$ is known.

THEOREM 7 [4]. *Given the sequence $\{U_m\}$ in* (18), *assume $u(y, a)$ is the solution to the PDE* (15) *and $p(x, a)$ is given in a homogeneous series as*

$$p(x, a) = \sum_{m=0}^{\infty} p_m(x) \frac{a^m}{m!}. \tag{23}$$

Let q be defined via the following composition in a homogeneous series:

$$q(y, a) = p(u(y, a), a) = \sum_{m=0}^{\infty} q_m(y) \frac{a^m}{m!}. \tag{24}$$

The sequence $\{q_m\}$ in (24) *satisfies*

$$q_i = p_0^{(i)}, \tag{25}$$

where, for $i = 0, 1, 2, \dots,$ the sequence $p_{i-m}^{(m)}$ is defined recursively via

$$p_i^{(0)} = p_i, \tag{26}$$

$$p_{i-m}^{(m)} = p_{i-m+1}^{(m-1)} + \sum_{j=0}^{i-m} \binom{i-m}{j} p_{i-m-j}^{(m-1)} * U_j, \qquad m = 1, 2, \dots, i. \tag{27}$$

Here

$$\binom{i-m}{j} = \frac{(i-m)!}{j!(i-m-j)!}$$

is the binomial coefficient and

$$f * g(x) := \frac{\partial f(x)}{\partial x} g(x)$$

is the Lie derivative.

A procedure for deriving an explicit expression for $u(y, a)$ from U is given in the following.

PROCEDURE 1. Setting $p(x, a) = U(x, a)$ in Theorem 7 yields an expansion for $q(y, a) = U(u(y, a), a) = \sum_{m=0}^{\infty} q_m(y)(a^m/m!)$ via (24). Substituting the homogeneous series expansions for $u(y, a)$ and $U(u(y, a), a)$ into the left-hand and right-hand sides of the PDE (15), respectively, gives

$$u_{m+1} = q_m, \qquad m = 0, 1, 2, 3, \dots,$$

where

$$u(y, a) = \sum_{m=0}^{\infty} u_m(y) \frac{a^m}{m!}. \tag{28}$$

The first few elements in the sequence $\{q_m\}$ can be written as follows:

$$u_1(y) = q_0(y) = U_0(y),$$

$$u_2(y) = q_1(y) = U_1(y) + \frac{\partial U_0(y)}{\partial y} U_0(y),$$

$$\begin{aligned} u_3(y) = q_2(y) = U_2(y) &+ \frac{\partial U_1(y)}{\partial y} U_0(y) + \frac{\partial U_0(y)}{\partial y} U_1(y) \\ &+ \frac{\partial q_1(y)}{\partial y} U_0(y), \end{aligned}$$

$$\begin{aligned} u_4(y) &= q_3(y) \\ &= U_3(y) + \frac{\partial U_2(y)}{\partial y} U_0(y) + 2\frac{\partial U_1(y)}{\partial y} U_1(y) \\ &\quad + \frac{\partial}{\partial y}\left(U_2(y) + \frac{\partial U_1(y)}{\partial y} U_0(y) + \frac{\partial U_0(y)}{\partial y} U_1(y) \right) U_0(y) \\ &\quad + \frac{\partial q_1(y)}{\partial y} U_1(y) + \frac{\partial q_2(y)}{\partial y} U_0(y) + \frac{\partial U_0(y)}{\partial y} U_2(y). \end{aligned}$$

In general, for $m = 1, 2, 3, \ldots,$

$$u_m = U_{m-1} + \Lambda_m, \tag{29}$$

where Λ_m is a map depending only on $U_0, U_1, \ldots, U_{m-2}$.

Thus, for a given sequence $\{u_m\}$, there exists a unique sequence $\{U_m\}$ satisfying the PDE (15) and vice versa.

The following lemma, which is proved in [4], will be used later.

Consider a homogeneous power series expansion for the forward transformation

$$v(x, a) = \sum_{m=0}^{\infty} v_m(x) \frac{a^m}{m!}. \tag{30}$$

LEMMA 8 [4]. *If, for* $m = 1, 2, \ldots,$ f_m *in* (16) *is a homogeneous polynomial map of degree* $m + 1$ *and* U_m *in* (18) *is a homogeneous polynomial map of degree* $m + 2$, *then* g_m, v_m, *and* u_m *in* (17), (28), *and* (30), *respectively, are homogeneous polynomial maps of degree of* $m + 1$.

3.2. *Normal Form Computation*

As was shown in Section 2, the normal form can be derived using a sequence of near identity transformations. Although it does demonstrate the concept of normal forms, this method for computing normal forms is very inefficient. This method is also not appropriate for implementation using a symbolic computation package such as Maple or Mathematica. However, with slight modifications, the algorithms introduced in Theorems 5 and 7 may be used to derive normal forms and the associated transformations using a symbolic computation package.

Assume the original nonlinear dynamical system is given by

$$\dot{x} = f(x) = \sum_{m=0}^{\infty} f^{m+1}(x) \frac{1}{m!}, \qquad x \in R^n, \tag{31}$$

where $f^1(x) = Ax$ with $A = Df(0)$, $f^m \in H(n, n, m)$.

The goal of this portion of the work is to obtain the transformation

$$x = u(\bar{x}) = \sum_{m=0}^{\infty} u^{m+1}(\bar{x}) \frac{1}{m!}, \qquad \bar{x} \in R^n, \tag{32}$$

where $u^1(\bar{x}) = \bar{x}$ and $u^m \in H(n, n, m)$, along with the transformation

$$\bar{x} = v(x) = \sum_{m=0}^{\infty} v^{m+1}(x) \frac{1}{m!}, \tag{33}$$

where $v^1(x) = x$ and $v^m \in H(n, n, m)$, so that the dynamics for $\bar{x}$ are in normal form given by

$$\dot{\bar{x}} = g(\bar{x}) \triangleq \frac{\partial v(x)}{\partial x} f(x)\bigg|_{x=u(\bar{x})} = \sum_{m=0}^{\infty} g^{m+1}(\bar{x}) \frac{1}{m!}, \tag{34}$$

where $g^1(\bar{x}) = A\bar{x}$ and $g^m \in \operatorname{Ker}(L_{A^T}^m)$. (In actuality, only transformations $u(\cdot) = \sum_{m=1}^{k} u^m(\cdot)(1/m!)$ and $v(\cdot) = \sum_{m=1}^{k} v^m(\cdot)(1/m!)$ which bring the system to an *approximate* kth-order normal form, $\dot{\bar{x}} = \sum_{m=1}^{k} g^m(\bar{x})(1/m!) + O(\|\bar{x}\|^{k+1})$ will be computed.)

Just as in Section 3.1, an artificial parameter that scales the variables according to $x \mapsto ax$, $\bar{x} \mapsto a\bar{x}$ is introduced here:

$$\dot{x} = \frac{f(ax)}{a} \triangleq f(x, a) = \sum_{m=0}^{\infty} f^{m+1}(x) \frac{a^m}{m!}, \tag{35}$$

$$x = \frac{u(a\bar{x})}{a} \triangleq u(\bar{x}, a) = \sum_{m=0}^{\infty} u^{m+1}(\bar{x}) \frac{a^m}{m!}, \tag{36}$$

$$\bar{x} = \frac{v(ax)}{a} \triangleq v(x, a) = \sum_{m=0}^{\infty} v^{m+1}(x) \frac{a^m}{m!}, \tag{37}$$

$$\dot{\bar{x}} = \frac{g(a\bar{x})}{a} \triangleq g(\bar{x}, a) = \sum_{m=0}^{\infty} g^{m+1}(\bar{x}) \frac{a^m}{m!}. \tag{38}$$

According Lemma 8, the PDE becomes

$$\frac{\partial u}{\partial a} = U(u, a) = \sum_{m=0}^{\infty} U^{m+2}(u) \frac{a^m}{m!}, \qquad u(\bar{x}, 0) = \bar{x}, \tag{39}$$

where $U^i(\cdot) \in H(n, n, i)$.

The process of computing the normal form dynamics can be accomplished recursively using Theorem 5 and Procedure 1 for the successive computation of $\{g^i(\cdot)\}$ and $\{u^i(\cdot)\}$. This procedure is summarized as follows:

PROCEDURE 2. Take f_m, g_m, and U_{m-1} in Theorem 5 to be f^{m+1}, g^{m+1}, and U^{m+1} in (35), (38), and (39), respectively. Beginning at step 1, with $m = 1$ and $g^1 = f^1$, each pair U^m and g^m is determined iteratively. At step m, a parameterized form for $U^m \in H(n, n, m)$ is assumed. Note that $g^1, \dots, g^{m-1}$ and $U^2, \dots, U^{m-1}$ are determined from previous steps. The homogeneous mapping g^m is then computed using the algorithm in Theorem 5. Restricting g^m to $\operatorname{Ker}(L_{A^T}^m)$ will fix some coefficients in U^m.

Other coefficients in U^m are free to be chosen. These free coefficients are simply set to zero for this work. Once all of the coefficients in U^m are determined, both u^m and g^m are also determined.

4. CONSTRUCTING LYAPUNOV FUNCTIONS

In [8], a general procedure for constructing normal form transformations that give rise to the construction of Lyapunov functions for critical nonlinear systems in normal form was provided. Lyapunov functions for Case 1 and Case 2 critical nonlinear systems were explicitly constructed. The problem with obtaining the functions in the original coordinates was rooted in a lack of a systematic computationally efficient method for obtaining the normal form transformations. In this section, Procedure 2 will be implemented to construct transformations that bring a system to a kth-order approximate normal form. These transformations will be used to construct Lyapunov functions for Case 1 and Case 2 critical nonlinear systems in their original coordinates.

A kth-order approximate normal form for a smooth critical nonlinear system (1), (2), has the form [9]

$$\begin{aligned}\dot{\bar{x}}_c &= A^0\bar{x}_c + \varphi(\bar{x}_c, \bar{x}_s) \\ &= A^0\bar{x}_c + \varphi_1(\bar{x}_c) + \cdots + \varphi_{k-1}(\bar{x}_c) + O\left(|\bar{x}_c, \bar{x}_s|^{k+1}\right),\end{aligned} \tag{40}$$

$$\begin{aligned}\dot{\bar{x}}_s &= A^-\bar{x}_s + \psi(\bar{x}_c, \bar{x}_s) \\ &= A^-\bar{x}_s + \psi_1(\bar{x}_c, \bar{x}_s) + \cdots + \psi_{k-1}(\bar{x}_c, \bar{x}_s) + O\left(|\bar{x}_c, \bar{x}_s|^{k+1}\right),\end{aligned} \tag{41}$$

where $\psi_i(\bar{x}_c, \bar{x}_s = 0) = 0$, $i = 1, 2, \ldots, k-1$.

Notice the local decoupling of the critical dynamics from the stable dynamics. The following results summarize work in [6, 7].

For an LAS Case 1 system, there exists an odd integer k such that $a_k < 0$ and

$$\dot{\bar{x}}_c = a_k\bar{x}_c^k + O\left(|\bar{x}_c, \bar{x}_s|^{k+1}\right). \tag{42}$$

Similarly, for an LAS Case 2 system, there exists an odd integer $k = 2m + 1$ such that $a_m < 0$ and

$$\begin{aligned}\dot{\bar{x}}_{c1} = {}& \omega\bar{x}_{c2} - \left(\bar{x}_{c1}^2 + \bar{x}_{c2}^2\right)b_1\bar{x}_{c2} - \cdots - \left(\bar{x}_{c1}^2 + \bar{x}_{c2}^2\right)^{m-1}b_{m-1}\bar{x}_{c2} \\ & - \left(\bar{x}_{c1}^2 + \bar{x}_{c2}^2\right)^m\left(b_m\bar{x}_{c2} - a_m\bar{x}_{c1}\right) + O\left(|\bar{x}_c, \bar{x}_s|^{(k+1)}\right),\end{aligned} \tag{43}$$

$$\begin{aligned}\dot{\bar{x}}_{c2} = {}& -\omega\bar{x}_{c1} + \left(\bar{x}_{c1}^2 + \bar{x}_{c2}^2\right)b_1\bar{x}_{c1} + \cdots + \left(\bar{x}_{c1}^2 + \bar{x}_{c2}^2\right)^{m-1}b_{m-1}\bar{x}_{c1} \\ & + \left(\bar{x}_{c1}^2 + \bar{x}_{c2}^2\right)^m\left(a_m\bar{x}_{c2} + b_m\bar{x}_{c1}\right) + O\left(|\bar{x}_c, \bar{x}_s|^{(k+1)}\right).\end{aligned} \tag{44}$$

Let $P = P^T > 0$ be a matrix such that $A^{-T}P + PA^{-} < 0$. The following theorem is proved in [8].

THEOREM 9. *$\overline{V}(\bar{x}_c, \bar{x}_s) = \bar{x}_c^2 + \bar{x}_s^T P \bar{x}_s$ is a Lyapunov function for the Case* 1 *system* (42), (41), *and $\overline{V}(\bar{x}_c, \bar{x}_s) = (\bar{x}_{c1}^2 + \bar{x}_{c1}^2) + \bar{x}_s^T P \bar{x}_s$ is a Lyapunov function for the Case* 2 *system* (41), (43), (44).

Suppose now that the system (40) and (41) is obtained from (1) and (2) via the polynomial transformation $x = u(\bar{x}) = \bar{x} + u^2(\bar{x}) + u^3(\bar{x})/2! + \cdots + u^k(\bar{x})/(k-1)!$. The associated forward transformation will have a homogeneous series expansion of the form $\bar{x} = v(x) = x + v^2(x) + v^3(x)/2! + \cdots$.

The following theorem can be proved from Theorems 5 and 7 and Lemma 8.

THEOREM 10. *For the relations* (31)–(34), *each g^m in the homogeneous expansion of g in* (34) *depends only on $f^1, \ldots, f^m$ and $u^2, \ldots, u^m$ or, equivalently, $v^2, \ldots, v^m$, and each u^m (or v^m) in the homogeneous expansions of $u(\cdot)$ or $v(\cdot)$ from* (32) *or* (33), *respectively, depends only on $U_0, \ldots, U_{m-1}$ in* (15).

This theorem implies that the smooth critical nonlinear system (1), (2) is related to the system (40), (41) by some transformation of the form $x = \bar{x} + \sum_{m=2}^{k} u^m(\bar{x})/m!$ (or, equivalently, there is a finite transformation representing the forward transformation, $\bar{x} = x + \sum_{m=2}^{k} v^m(x)/m!$).

The following theorem follows naturally from the one above.

THEOREM 11. *For a smooth LAS critical nonlinear system of the form* (1), (2), *if $\overline{V}(\bar{x}_c, \bar{x}_s)$ is a Lyapunov function for the system* (40), (41), *then*

$$V(x_c, x_s) = \overline{V}\left(\bar{x} = (\bar{x}_c, \bar{x}_s) = x + \sum_{m=1}^{k-1} v_m(x)/m!\right)$$

is a Lyapunov function for the system in its original coordinates (1), (2). *For a Case* 1 *or Case* 2 *system, $\overline{V}(\bar{x}_c, \bar{x}_s)$ may be obtained from Theorem* 9.

The sequence v^m in (33) may be computed from Theorem 7 by taking $p(x, a) = v(x)$ as follows:

$$v^1(\bar{x}) = \bar{x},$$

$$v^2(\bar{x}) = -U_0(\bar{x}),$$

$$v^3(\bar{x}) = -\left(\frac{\partial v^2(\bar{x})}{\partial \bar{x}} U_0(\bar{x}) + \frac{\partial v^1(\bar{x})}{\partial \bar{x}} U_1(\bar{x})\right),$$

$$v^4(\bar{x}) = -\left(\frac{\partial v^3(\bar{x})}{\partial \bar{x}} U_0(\bar{x}) + 2\frac{\partial v^2(\bar{x})}{\partial \bar{x}} U_1(\bar{x}) + \frac{\partial v^1(\bar{x})}{\partial \bar{x}} U_2(\bar{x})\right),$$

$$v^5(\bar{x}) = -\left(\frac{\partial v^4(\bar{x})}{\partial \bar{x}} U_0(\bar{x}) + 3\frac{\partial v^3(\bar{x})}{\partial \bar{x}} U_1(\bar{x}) + 3\frac{\partial v^2(\bar{x})}{\partial \bar{x}} U_2(\bar{x}) + \frac{\partial v^1(\bar{x})}{\partial \bar{x}} U_3(\bar{x})\right),$$

$$v^6(\bar{x}) = -\left(\frac{\partial v^5(\bar{x})}{\partial \bar{x}} U_0(\bar{x}) + 4\frac{\partial v^4(\bar{x})}{\partial \bar{x}} U_1(\bar{x}) + 6\frac{\partial v^3(\bar{x})}{\partial \bar{x}} U_2(\bar{x}) + 4\frac{\partial v^2(\bar{x})}{\partial \bar{x}} U_3(\bar{x}) + \frac{\partial v^1(\bar{x})}{\partial \bar{x}} U_4(\bar{x})\right).$$

Using Procedure 2, the computation of the sequence $\{v^m\}$ is given as above. In general, it is much simpler than that of the sequence $\{u^m\}$, due to the simple structure of the critical dynamics in the normal form.

PROCEDURE 3. In general, v^m satisfies

$$v^{m+2}(\bar{x}) = = -\sum_{i=1}^{m+1} \binom{m}{i-1} \frac{\partial v^{m-i}(\bar{x})}{\partial \bar{x}} U_{i-1}(y). \tag{45}$$

5. SUMMARY

This paper presents a systematic procedure for constructing Lyapunov functions for a class of nonlinear systems, namely, locally asymptotically stable smooth critical nonlinear systems whose linearizations have only either one simple zero eigenvalue or a simple pair of purely imaginary eigenvalues, while all other eigenvalues are stable (have negative real parts). This procedure provides the capability to obtain a transformation that places the system in approximate normal form. Once such a system is in an appropriate (approximate) normal form, the local asymptotic stability of the system is easily determined and a Lyapunov function for the system can be found from the method presented in [8]. The Lyapunov function for the system in its original coordinates is obtained by substituting an appropriate transformation. Procedures that allow for the computation of normal forms and the associated transformations have also been introduced. The method presented here for Lyapunov function construction could be applied to more general than Case 1 and Case 2 critical nonlinear systems, provided a Lyapunov function for the system in approximate normal form could be found.

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